Exponential stability of stochastic theta method for nonlinear stochastic Volterra integro-differential equations

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1. Introduction

Integro-differential equations are very important in the study of many phenomena in physics, mechanics, medical, finance, sociology, biology, etc.

Volterra integro-differential equation was originated from predator-prey equations (Lotka-Volterra system)

$$dN_1 = \varepsilon_1 N_1 dt - \gamma_1 N_1 N_2 dt$$

$$dN_2 = \varepsilon_2 N_2 dt - \gamma_2 N_1 N_2 dt$$

where $\varepsilon_i > 0$ and $\gamma_i > 0$.

1. Introduction

These equations could be transformed to the following 2-dimensional integro-differential equation.

$$X' = AX + \int_0^t C(t-s)X(s)ds + \Gamma(t)$$

1. Introduction

This equation can be generalized to the following deterministic nonlinear Volterra integro-differential equation

$$x'(t) = f\left(t, x(t), \int_0^t G(t-s)x(s)ds\right) \tag{1}$$

1. Introduction

In 2014, P. Hu and C. Huang published a paper [**The stochastic θ-method for nonlinear stochastic Volterra integro-differential equations, Abstract and Applied Analysis, ID 583930, 13 pages**].

They considered the following stochastic Volterra integro-differential equation with convolution kernels

$$\begin{aligned} dx(t) &= f\left(x(t), \int_0^t G(t-s)x(s)ds\right) dt \\ &\quad + g\left(x(t), \int_0^t H(t-s)x(s)ds\right) dW(t), x(0) = x_0 \end{aligned} \tag{2}$$

1. Introduction

The authors considered in this paper the mean square exponential stability of the exact solution to the above stochastic Volterra integro-differentia equation under some given conditions, moreover, mean square convergence and mean square stability of the corresponding stochastic θ-method are also investigated.

We will only consider in this talk the mean square stability of both the exact solution and the corresponding stochastic θ-method, and we improved the conclusion of Hu and Huang. Roughly speaking, we will obtain stronger results under weaker conditions (compared with Hu and Huang).

2. Framework

Consider the following stochastic Volterra integro-differential equation

$$
\begin{aligned}
dx(t) = f\left(x(t), \int_0^t G(t-s)x(s)ds\right) dt \\
+ g\left(x(t), \int_0^t H(t-s)x(s)ds\right) dW(t), x(0) = x_0
\end{aligned} \tag{3}
$$

Suppose $f(0,0) = 0$, $g(0,0) = 0$, which implies that $X \equiv 0$ is the trivial solution of the above equation.

2. Framework

The corresponding θ-EM method (stochastic θ-method) of stochastic Volterra integro-differential equation is defined as the following.

$$\begin{aligned} X_{n+1} &:= X_n + h[\theta f(X_{n+1}, Z_{n+1}) + (1-\theta) f(X_n, Z_n)] \\ &\quad + g(X_n, \bar{Z}_n)\Delta W_n \end{aligned} \tag{4}$$

where $\theta \in [0, 1]$, h is the stepsize, $\Delta W_n = W((n+1)h) - W(h)$ is the increment of Brownian motion W.

$$Z_n := h \sum_{j=0}^{n-1} G((n-j)h)x_j, \quad \bar{Z}_n := h \sum_{j=0}^{n-1} G((n-j)h)x_j$$

2. Framework

Mean square exponential stability:
We say the solution to equation (3) is mean square exponentially stable if there exists $\lambda > 0$ such that

$$\limsup_{t\to\infty} \frac{\log \mathbb{E}|x(t)|^2}{t} \leq -\lambda.$$

Similarly, we say the θ-EM method (4) is mean square exponentially stable if there exists $\lambda > 0$ such that

$$\limsup_{k\to\infty} \frac{\log \mathbb{E}|X_k|^2}{kh} \leq -\lambda.$$

2. Framework

Almost sure exponential stability:
If

$$\limsup_{t\to\infty} \frac{\log |x(t)|^2}{t} \leq -\lambda, \ a.s.$$

or

$$\limsup_{k\to\infty} \frac{\log |X_k|^2}{kh} \leq -\lambda, \ a.s.,$$

we say the solution to equation (3) or (4) almost sure exponentially stable, respectively.

3. Known results

Suppose there exist six positive constants $\lambda_i, i = 1, ..., 4,\ \beta$ and γ such that

$$2x^T f(x, 0) \leq -\lambda_1 |x|^2, \tag{5}$$

$$|f(x, y) - f(x, 0)| \leq \lambda_2 |y|, \tag{6}$$

$$|g(x, y)|^2 \leq \lambda_3 |x|^2 + \lambda_4 |y|^2 \tag{7}$$

for all $x, y \in \mathbb{R}^d$, and

$$|G(t)| \vee |H(t)| \leq \beta e^{-\gamma t}, \quad t \geq 0. \tag{8}$$

3. Known results

Theorem 1 Assume that conditions (5)-(8) hold. If

$$-\lambda_1 + \lambda_2 + \lambda_3 + (\lambda_2 + \lambda_4)\frac{\beta^2}{\gamma^2} < 0, \tag{9}$$

then for any given initial data x_0, there exists a pair of positive constants ρ and C such that

$$\mathbb{E}|x(t)|^2 \leq C\mathbb{E}|x_0|^2 e^{-\rho t}, \ \forall t \geq 0,$$

i.e. the exact solution of (3) is mean square exponentially stable.

3. Known results

Theorem 2 Under conditions (5)-(8), if $\frac{1}{2} \leq \theta \leq 1$, then the stochastic θ method (4) is mean square asymptotically stable for any stepsize $h > 0$ (i.e. $\lim_{n\to\infty} \mathbb{E}|X_n|^2 = 0$ for any initial x_0).

4. Our results

4.1 Mean square exponential stability of the exact solution

Consider the following **local Lipschitz condition**: For any $R > 0$ and $|x| \vee |x'| \vee |y| \vee |y'| \leq R$, there exists $L_R > 0$ such that

$$|f(x, y) - f(x', y')| \vee |g(x, y) - g(x', y')| \leq L_R(|x - x'| + |y - y'|), \quad (10)$$

exponential decay condition: There exist $\beta > 0$ and $\gamma > 0$ such that

$$|G(t)| \vee |H(t)| \leq \beta e^{-\gamma t} \quad (11)$$

and **Khasminskii-type condition**

$$2\langle x, f(x, y)\rangle + |g(x, z)|^2 \leq -C_1|x|^2 + C_2|y|^2 + C_3|z|^2, \quad (12)$$

where $C_i, i = 1, 2, 3$ are positive constants.

4. Our results

Theorem 3

Assume (10), (11) and (12) hold. If $C_1 > \frac{\beta^2}{\gamma^2}(C_2 + C_3)$, then there exist a unique global solution to equation (3). Moreover, exact solution is mean square exponentially stable and almost surely exponentially stable.

4. Our results

4.2 Mean square exponential stability of the θ-EM method

To make sure the θ-EM method is well defined, we need the following one-sided Lipschitz conditon of f, i.e. There exists $L > 0$ such that

$$\langle x_1 - x_2, f(x_1, y) - f(x_2, y) \rangle \leq L|x_1 - x_2|^2, \forall x_1, x_2, y \in \mathbb{R}^d \qquad (13)$$

Theorem 4 Suppose all conditions in Theorem 3 hold with $C_1 > \frac{\beta^2}{\gamma^2}(C_2 + C_3)$. If the one-sided Lipschitz condition (13) also holds, then the θ-EM method (4) is mean square exponentially stable for $\theta \in (\frac{1}{2}, 1]$, and therefore, it is almost surely exponentially stable .

Comparison of the results

We obtain mean square exponential stability of the θ-EM method (4) in Theorem 4, while Hu and Huang only prove the mean square asymptotic stability in Theorem 2.

Moreover, our conditions is weaker than those of Hu and Huang. Indeed, conditions (5)-(7) imply condition (12). On the other hand, we don't need the linearity of g.

Sketch of the proof of Theorem 3

1. Using Itô's formula to $e^{\lambda t}|x(t)|^2$;
2. Using the fact that

$$\left|\int_0^s G(s-r)x(r)dr\right|^2 \vee \left|\int_0^s H(s-r)x(r)dr\right|^2 \le \frac{\beta^2}{\gamma^2}\int_0^s e^{-\gamma(s-r)}|x(r)|^2 dr$$

3. Choosing $\lambda < \gamma$. Then

$$e^{\lambda t}\mathbb{E}|x(t)|^2 \le \mathbb{E}|x_0|^2 + \left[(\lambda - C_1) + \frac{\beta^2}{\gamma(\gamma-\lambda)}(C_2 + C_3)\right]\int_0^t e^{\lambda s}\mathbb{E}|x(s)|^2 ds$$

4. Mean square exponential stability is a direct result by choosing $\lambda > 0$ small enough such that

$$(\lambda - C_1) + \frac{\beta^2}{\gamma(\gamma - \lambda)}(C_2 + C_3) \leq 0$$

5. Almost sure exponential stability could be obtained by continuous semimartingale convergence theorem (see the following slide).

Continuous semimartingale convergence theorem

Let $A(t), U(t)$ be two continuous $\mathscr{F}_t$ adapted increasing processes on $t \geq 0$ with $A(0) = U(0) = 0$ a.s. Let $M(t)$ be a real-valued continuous local martingale with $M(0) = 0$ a.s. Let ξ be a nonnegative $\mathscr{F}_0$-measurable random variable. Assume that $\{X(t)\}$ is a nonnegative semimartingale with the Doob-Meyer decomposition

$$X(t) = \xi + A(t) - U(t) + M(t), t \geq 0.$$

If $\lim_{i\to\infty} A(t) < \infty$ a.s., then

$$\lim_{t\to\infty} X(t) < \infty \text{ and } \lim_{t\to\infty} U(t) < \infty, \quad a.s.$$

Sketch of the proof of Theorem 4

The idea is based on [Exponential stability of the exact solutions and θ-EM approximations to neutral SDDEs with Markov switching, J. Comput. Appl. Math., 2015, 285, 230-242.]

1. Define $F_k := X_k - \theta h f(X_k, Z_k)$. Prove that for $0 < C < C_1$, we can choose sufficiently small $h > 0$ such that

$$|F_{k+1}|^2 \leq |F_k|^2 - Ch|F_k|^2 + (C_2|Z_k|^2 + C_3|\bar{Z}_k|^2)h + M_k.$$

Sketch of the proof of Theorem 4

2. Using the fact that

$$|F_k|^2 \geq |X_k|^2 - \theta h(-C_1|X_k|^2 + C_2|Z_k|^2 + C_3|\bar{Z}_k|^2)$$

and that

$$|Z_k|^2 \vee |\bar{Z}_k|^2 \leq \frac{\beta^2}{\gamma} h \sum_{j=0}^{k-1} e^{-\gamma(n-j)h}|X_j|^2$$

3. We can choose $A > 1$ such that $1 - Ch - A^{-h} < 0$, then

$$A^{kh}\mathbb{E}|F_k|^2 \leq \mathbb{E}|F_0|^2.$$

Sketch of the proof of Theorem 4

4. Then

$$A^{kh}\mathbb{E}|X_k|^2 \leq \frac{\mathbb{E}|X_0|^2}{1 + C_1\theta h} + \frac{\theta h^2\beta^2(C_2 + C_3)}{\gamma(1 + C_1\theta h)}\sum_{j=0}^{k-1}(Ae^{-\gamma})^{(k-j)h}A^{jh}\mathbb{E}|X_j|^2$$

5. Complete induction yields the required results.
6. Almost sure exponential stability could be obtained by discrete semimartingale convergence theorem or by using Chebyshev's inequality and Borel-Cantelli lemma.

5. An example

Consider the following scalar stochastic Volterra integro-differential equation

$$
\begin{aligned}
dx(t) = &\left(-6x(t) - x^5(t) + \int_0^t e^{-(t-s)}x(s)ds\right) dt \\
&+ \left[2\frac{x^3(t)\int_0^t e^{-2(t-s)}x(s)ds}{1+\left(\int_0^t e^{-2(t-s)}x(s)ds\right)^2} + \int_0^t e^{-2(t-s)}x(s)ds\right] dW_t
\end{aligned}
\tag{14}
$$

with the initial value $x_0 = 1$.

It is clear that the coefficients $f(x,y) = -(6x + x^5 - y)$ and $g(x,y) = 2\frac{x^3 y}{(1+y^2)} + y$ satisfy the local Lipschitz condition (10). (11) holds for $G(t) = e^{-t}$ and $H(t) = e^{-2t}$ with $\beta = \gamma = 1$. And f satisfies one-sided Lipschitz condition (13).

Moreover, condition (12) holds for $C_1 = 11, C_2 = 1, C_3 = 8$. And $C_1 > \frac{\beta^2}{\gamma^2}(C_2 + C_3)$ holds with $\beta = \gamma = 1$.

Then by Theorem 3 and by Theorem 4, we know that the exact solution to equation (1) is mean square exponentially stable. Moreover, one can choose sufficiently small stepsize h such that the θ-EM method ($\theta \in (\frac{1}{2}, 1]$) is also mean square exponentially stable.

Thanks for your attention!

谢 谢 大 家!

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